General Form of Color Charge of the Quark

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Abstract

In Maxwell theory the the constant electric charge e of the electron is consistent with the continuity equation $\partial_\mu j^\mu(x) = 0$ where $j^\mu(x)$ is the current density of the electron. However, in Yang-Mills theory the Yang-Mills color current density $j^{\mu a}(x)$ of the quark satisfies the equation $D_\mu[A]j^{\mu a}(x) = 0$ which is not a continuity equation ($\partial_\mu j^{\mu a}(x) \neq 0$) which implies that the color charge of the quark is not constant. Since the charge density of a point particle can be obtained from the quantum wave function of that particle, one finds that the charge density of a point particle may depend on space-coordinate $\vec{x}$. However, since the charge of a point particle is obtained from the corresponding charge density after integrating over the entire (physically) allowed volume $V = \int d^3\vec{x}$, one finds that the charge of the point particle can not depend on space-coordinate $\vec{x}$. Since the color charge of the quark is not constant and it can not depend on space coordinate $\vec{x}$, one finds that the color charge $q^a(t)$ of the quark in Yang-Mills theory has to be time dependent. In this paper we derive the general form of eight time dependent fundamental color charges $q^a(t)$ of a quark in Yang-Mills theory in SU(3) where $a = 1, 2, ...8$.

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I. INTRODUCTION

The electric and magnetic phenomena in nature originates from the electric charge. The electric charge of an electron is a fundamental quantity of nature which is constant and has the experimentally measured value $e = 1.6 \times 10^{-19}$ coulombs. Note that there are two types of electric charges in nature, 1) positive (+) charge and 2) negative (-) charge. The charge of the electron is negative (-). The electric charge produces electromagnetic force in the nature.

However, inside a hadron (such as proton and neutron) the electromagnetic force can not bind the quarks together. The force which is responsible for bound state hadron formation is called the strong force or color force which is a fundamental force of nature. This strong force or color force is not produced from the electric charge of the quark but it is produced from the color charge of the quark. The color charge is a fundamental charge of nature which exists inside hadrons.

Note that in Maxwell theory in electrodynamics the constant electric charge e of the electron is consistent with the continuity equation

$$\partial_\mu j^\mu(x) = 0 \tag{1}$$

where $j^\mu(x)$ is the electric current density of the electron. However, in Yang-Mills theory the Yang-Mills color current density $j^{\mu a}(x)$ of the quark satisfies the equation [1, 2]

$$D_\mu[A] j^{\mu a}(x) = 0, \qquad D_\mu^{ab}[A] = \delta^{ab}\partial_\mu + g f^{acb} A_\mu^c(x), \qquad a, b, c = 1, 2...8 \tag{2}$$

which is not a continuity equation ($\partial^\mu j_\mu^a(x) \neq 0$) which implies that the color charge of the quark is not constant.

Also, it is important that the conserved color charges are not directly observable – only color representations – because of the unbroken gauge invariance of QCD. Thus, the concept of constant color charge seems unphysical. The form of the color charge of the quark can be obtained from the zero component of the color current density $j_0^a(x)$ of the quark. For earlier works on classical Yang-Mills theory, see [3–6].

In quantum mechanics the charge density and probability density of the point particle can be obtained from the quantum wave functions. Hence one finds that the charge density is more microscopic and may depend on the space coordinate $\vec{x}$. For example, consider charge

and charge density in a volume $V = \int d^3\vec{x}$. The total charge in the volume $V = \int d^3\vec{x}$ can be obtained from the corresponding charge density after integrating over the entire (physically) allowed volume $V = \int d^3\vec{x}$. Hence one finds that the total charge in the volume $V = \int d^3\vec{x}$ is independent of space coordinate $\vec{x}$. If there is only one point charge in the entire (physically) allowed volume $V = \int d^3\vec{x}$ then one finds that the charge of the point particle is independent of the space coordinate $\vec{x}$.

This implies that if the color charge of the quark is not constant then it has to be time dependent.

The color current density in eq. (2) is related to the quark field $\psi_i(x)$ via the equation [1, 2]

$$j^a_\mu(x) = g\bar{\psi}_i(x)\gamma_\mu T^a_{ij}\psi_j(x), \qquad a = 1, 2, ...8; \qquad i, j = 1, 2, 3. \tag{3}$$

Since the color current density $j^{\mu a}(x)$ of a quark in eq. (3) has eight color components $a = 1, 2, ...8$ one finds that there are eight time dependent color charges of a quark.

We denote eight time dependent fundamental color charges of a quark by $q^a(t)$ where $a = 1, 2, ...8$ are color indices. These eight time dependent fundamental color charges $q^a(t)$ of a quark are independent of quark flavor, *i.e.*, a color charge $q^a(t)$ of the u (up) quark is same as that of d (down), S (strange), c (charm), B (bottom) or t (top) quark.

It is useful to remember that the indices i=1,2,3=RED, BLUE, GREEN are not color charges of the quark but they are color indices of the quark field $\psi_i(x)$ in eq. (3). Color charges $q^a(t)$ of a quark are functions and they have values. This is analogous to electric charge '$-e$' of the electron which is not just a '$-$' sign but it has a constant value $e = 1.6 \times 10^{-19}$ coulombs. Similarly RED, BLUE, GREEN symbols are not color charges of a quark but $q^a(t)$ are color charges of a quark where $a = 1, 2, ..8$. Hence one finds that a quark does not have three color charges but a quark has eight color charges. In Maxwell theory the electric charge e produces electromagnetic force and in Yang-Mills theory the color charges $q^a(t)$ produce color force or strong force.

In SU(2) Yang-Mills theory the color current density is given by

$$j^i_\mu(x) = g\bar{\psi}_k(x)\gamma_\mu\tau^i_{kn}\psi_n(x), \qquad i = 1, 2, 3; \qquad k, n = 1, 2 \tag{4}$$

where τ^i are three $2 \otimes 2$ Pauli matrices. Hence one finds that there are three color charges $q_i(t)$ of a fermion in Yang-Mills theory in SU(2) where $i = 1, 2, 3$. We find that the general

form of three time dependent fundamental color charges of a fermion in Yang-Mills theory in SU(2) is given by

$$q_1(t) = g \times \sin\theta(t) \times \cos\phi(t),$$
$$q_2(t) = g \times \sin\theta(t) \times \sin\phi(t),$$
$$q_3(t) = g \times \cos\theta(t) \tag{5}$$

where the time dependent real phases $\theta(t)$ and $\phi(t)$ can not be independent of time t and the allowed range of $\theta(t)$, $\phi(t)$ are given by

$$\frac{\pi}{3} \le \theta(t) \le \frac{2\pi}{3}, \qquad -\pi \le \phi(t) \le \pi. \tag{6}$$

[See eqs. (121) and (122) for the derivation of eqs. (5) and (6)].

In this paper we will derive the general form of eight time dependent fundamental color charges $q^a(t)$ of a quark in Yang-Mills theory in SU(3) where $a = 1, 2, ...8$. We find that the general form of eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) is given by

$$q_1(t) = g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \cos\phi_{12}(t),$$
$$q_2(t) = g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \sin\phi_{12}(t),$$
$$q_3(t) = g \times \cos\theta(t) \times \sin\phi(t)$$
$$q_4(t) = g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \cos\phi_{13}(t),$$
$$q_5(t) = g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \sin\phi_{13}(t),$$
$$q_6(t) = g \times \sin\theta(t) \times \cos\sigma(t) \times \cos\phi_{23}(t),$$
$$q_7(t) = g \times \sin\theta(t) \times \cos\sigma(t) \times \sin\phi_{23}(t),$$
$$q_8(t) = g \times \cos\theta(t) \times \cos\phi(t) \tag{7}$$

where

$$\sin^{-1}(\sqrt{\frac{2}{3}}) \le \theta(t) \le \pi - \sin^{-1}(\sqrt{\frac{2}{3}}), \qquad 0 \le \sigma(t),\ \eta(t) \le \frac{\pi}{2},$$
$$0 \le \phi(t) \le 2\pi, \qquad -\pi \le \phi_{12}(t),\ \phi_{13}(t),\ \phi_{23}(t) \le \pi. \tag{8}$$

Hence we find that the general form of eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) is given by eq. (7) where the ranges of real time

dependent phase factors $\theta(t),\ \sigma(t),\ \eta(t),\ \phi(t),\ \phi_{12}(t),\ \phi_{13}(t),\ \phi_{23}(t)$ are given by eq. (8). Note that the real time dependent phase factors $\theta(t),\ \sigma(t),\ \eta(t),\ \phi(t),\ \phi_{12}(t),\ \phi_{13}(t),\ \phi_{23}(t)$ in eq. (7) can not be independent of time t because if they are independent of t then the Yang-Mills potential $A^{\mu a}(x)$ reduces to Maxwell-like potential $A^{\mu}(x)$.

We will provide derivations of eqs. (7) and (8) in this paper.

The paper is organized as follows. In section II we discuss exact form of abelian pure gauge potential produced by the electron in Maxwell theory. In section III we discuss charge and charge density of a point particle using quantum mechanics. In section IV we show that color charges of a quark in Yang-Mills theory are time dependent. In section V we discuss relation between coupling constant and color charge in Yang-Mills theory. In section VI we discuss color current density and color charge of quark and general form of Yang-Mills potential. In section VII we discuss fermion color current density, fermion wave function and Pauli Matrices in Yang-Mills theory in SU(2). In section VIII we derive general form of three time dependent fundamental color charges $q_i(t)$ of a fermion in Yang-Mills theory in SU(2) where $i = 1, 2, 3$. In section IX we discuss Yang-Mills color current density, quark wave function and Gell-Mann Matrices in Yang-Mills theory in SU(3). In section X we derive general form of eight time dependent fundamental color charges $q^a(t)$ of a quark in Yang-Mills theory in SU(3) where $a = 1, 2, ...8$. We conclude in section XI.

II. ABELIAN PURE GAUGE POTENTIAL IN MAXWELL THEORY

The Maxwell equation in electrodynamics is given by [7]

$$\partial_\nu F^{\nu\mu} = j^\mu, \qquad\qquad \partial_\mu F_{\nu\beta} + \partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu} = 0 \tag{9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10}$$

In classical electrodynamics if the electric charge e is of a point particle whose position in the inertial frame K is $\vec{X}(t)$ then the charge density $\rho(t,\vec{x})$ and current density $\vec{j}(t,\vec{x})$ of the point charge e in that frame are given by [7]

$$\rho(t,\vec{x}) = j_0(t,\vec{x}) = e\ \delta^{(3)}(\vec{x} - \vec{X}(t)),$$
$$\vec{j}(t,\vec{x}) = e\ \vec{v}(t)\ \delta^{(3)}(\vec{x} - \vec{X}(t)) \tag{11}$$

where

$$\vec{v}(t) = \frac{d\vec{X}(t)}{dt} \tag{12}$$

is the charge's velocity in that frame K. From eq. (11) one finds

$$\frac{\partial j_0(t,\vec{x})}{\partial t} = -e\ v_x(t)\ \delta'(x-X(t))\ \delta(y-Y(t))\ \delta(z-Z(t))$$
$$-e\ v_y(t)\ \delta(x-X(t))\ \delta'(y-Y(t))\ \delta(z-Z(t)) - e\ v_z(t)\ \delta(x-X(t))\ \delta(y-Y(t))\ \delta'(z-Z(t)) \tag{13}$$

where

$$\delta'(w) = \frac{d[\delta(w)]}{dw} \tag{14}$$

and

$$\vec{\nabla}\cdot\vec{j}(t,\vec{x}) = e\ v_x(t)\ \delta'(x-X(t))\ \delta(y-Y(t))\ \delta(z-Z(t))$$
$$+e\ v_y(t)\ \delta(x-X(t))\ \delta'(y-Y(t))\ \delta(z-Z(t)) + e\ v_z(t)\ \delta(x-X(t))\ \delta(y-Y(t))\ \delta'(z-Z(t)). \tag{15}$$

From eqs. (13) and (15) one finds

$$\partial_\mu j^\mu(x) = \frac{\partial j^\mu(x)}{\partial x^\mu} = \frac{\partial j_0(t,\vec{x})}{\partial t} + \vec{\nabla}\cdot\vec{j}(t,\vec{x}) = 0 \tag{16}$$

which is the continuity equation in Maxwell theory. Hence one finds that the constant charge e of the electron satisfies the continuity eq. (16).

In the covariant formulation the current density of the electron of charge e is given by [7]

$$j^\mu(x) = e\int d\tau\ u^\mu(\tau)\ \delta^{(4)}(x-X(\tau)) \tag{17}$$

which satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0 \tag{18}$$

where

$$u^\mu(\tau) = \frac{dX^\mu(\tau)}{d\tau} \tag{19}$$

is the four-velocity of the electron and $X^\mu(\tau)$ four-coordinate of the electron.

The solution of the inhomogeneous wave equation

$$\partial_\nu \partial^\nu A^\mu(x) = j^\mu(x). \tag{20}$$

is given by

$$A^\mu(x) = \int d^4x' D_r(x - x') j^\mu(x') \tag{21}$$

where $D_r(x - x')$ is the retarded Greens function [7]. From eqs. (17) and (21) we find

$$A^\mu(x) = e \frac{u^\mu(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))} \tag{22}$$

which satisfies the Lorentz gauge condition

$$\partial_\mu A^\mu(x) = 0 \tag{23}$$

where τ_0 is obtained from the solution of the retarded equation

$$x_0 - X_0(\tau_0) = |\vec{x} - \vec{X}(\tau_0)|. \tag{24}$$

In eq. (22) the four-vector x^μ is the time-space position at which the electromagnetic field is observed and the four-vector $X^\mu(\tau_0)$ is the time-space position of the electron where the electromagnetic field was produced.

Hence we find that if $A^\mu(x)$ satisfies Lorentz gauge condition $\partial_\mu A^\mu(x) = 0$ then eq. (20) reduces to the inhomogeneous Maxwell equation

$$\partial_\nu F^{\nu\mu}(x) = j^\mu(x) \tag{25}$$

where

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{26}$$

For $x^\mu \neq X^\mu(\tau)$ we find from eq. (22) and (26) that

$$\partial_\nu F^{\nu\mu}(x) = 0 \tag{27}$$

which satisfies Maxwell equation given by eq. (25) where $j^\mu(x)$ is given by eq. (17).

Note that an electron has non-zero mass which implies that an electron can not travel exactly at speed of light $v = c$. When the electron in uniform motion is at its highest speed (which is arbitrarily close to the speed of light $v \sim c$) we find from eq. (22)

$$A^\mu(x) = e \frac{\beta^\mu_{\sim c}}{\beta_{\sim c} \cdot (x - X(\tau_0))}, \qquad \beta^\mu_{\sim c} = (1,\ \vec{\beta}_{\sim c}), \qquad \vec{\beta}^2_{\sim c} = \frac{v^2}{c^2} \sim 1. \tag{28}$$

From eqs. (28) and (26) we find that

$$F^{\mu\nu}(x) \sim 0, \qquad \text{when} \qquad v \sim c. \tag{29}$$

From eqs. (29), (28) and (26) we find that

$$A^\mu = e\frac{\beta^\mu_{\sim c}}{\beta_{\sim c}\cdot(x - X(\tau_0))} \sim \partial^\mu \omega(x), \qquad \text{where} \qquad \omega(x) = e\,\ln[\beta_{\sim c}\cdot(x - X(\tau_0))]. \tag{30}$$

From eq. (30) we find that when the electron in uniform motion is at its highest speed (which is arbitrarily close to the speed of light $v \sim c$) then the electromagnetic potential produced from this electron becomes an (approximate) pure gauge potential $A^\mu \sim \partial^\mu \omega(x)$. We call the electromagnetic potential $A^\mu(x)$ in eq. (30) as (approximate) pure gauge potential because an electron has non-zero mass (even if it is very small) and hence it can not travel exactly at speed of light $\beta^\mu_c = (1,\ \vec{\beta}_c)$, $\vec{\beta}^2_c = \frac{v^2}{c^2} = 1$ to produce exact pure gauge potential

$$A^\mu(x) = e\frac{\beta^\mu_c}{\beta_c\cdot(x - X(\tau_0))} = \partial^\mu \omega(x), \qquad \omega(x) = e\,\ln[\beta_c\cdot(x - X(\tau_0))]. \tag{31}$$

From eq. (30) one finds that the $\omega(x)$ appearing in the pure gauge potential

$$A^\mu(x) = \partial^\mu \omega(x) \tag{32}$$

in U(1) Maxwell theory is linearly proportional to e, *i.e.*,

$$\omega(x) \propto e. \tag{33}$$

Similarly one finds that the $\omega^a(x)$ appearing in the abelian-like non-abelian pure gauge potential

$$\mathcal{A}^{\mu a}(x) = \partial^\mu \omega^a(x) \tag{34}$$

in Yang-Mills theory is linearly proportional to g, *i.e.*,

$$\omega^a(x) \propto g \tag{35}$$

whereas the SU(3) pure gauge potential $A^{\mu a}(x)$ in

$$T^a A^{\mu a}(x) = \frac{1}{ig}[\partial^\mu U(x)]U^{-1}(x), \qquad U(x) = e^{igT^a\omega^a(x)} \tag{36}$$

in SU(3) Yang-Mills theory is non-linear function of g. Note that eq. (36) contains infinite number of terms up to infinite powers of g and/or up to infinite powers $\omega^a(x)$. Eq. (34) is only the first term in the expansion of $A^{\mu a}(x)$ in eq. (36).

III. CHARGE AND CHARGE DENSITY OF A POINT PARTICLE USING QUANTUM MECHANICS

In quantum mechanics the electron is described by the wave function $\psi(x)$. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[\gamma_\mu \partial^\mu - m + e\gamma_\mu A^\mu(x)]\psi \tag{37}$$

from which we obtain the Dirac equation of the electron

$$[\gamma_\mu \partial^\mu - m + e\gamma_\mu A^\mu(x)]\psi(x) = 0. \tag{38}$$

From eq. (37) we find that the current density of the electron is given by

$$j^\mu(x) = e\bar{\psi}(x)\gamma^\mu \psi(x) \tag{39}$$

which satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0. \tag{40}$$

Note that the field $\psi(x)$ in eqs. (38) and (39) is the Schrodinger wave function which is yet to be quantized in the sense of second quantization (field quantization) [2].

Note that in classical mechanics the charge density of a point charge may be described by delta function distribution, see eq. (11), which implies that the charge density at the position $\vec{x} \neq \vec{X}(t)$ is zero where $\vec{X}(\tau)$ is the spatial position of the electron and $\vec{x}$ is the spatial position at which the current density is determined. However, in quantum mechanics the probability density $\psi^\dagger(t,\vec{x})\psi(t,\vec{x})$ of finding a particle is not defined at a fixed position $\vec{x} = \vec{X}(t)$ but rather it is defined in a certain (infinitesimal) volume element. Hence one finds that in quantum mechanics the charge density $e\psi^\dagger(t,\vec{x})\psi(t,\vec{x})$ is not zero when $\vec{x} \neq \vec{X}(t)$.

However, when integrated over the entire (physically) allowed volume $V = \int d^3\vec{x}$, the charge density in quantum mechanics and the charge density in classical mechanics reproduce the same charge e of the electron, *i.e.*, for the normalized wave function

$$\int d^3\vec{x}\ \psi^\dagger(x)\psi(x) = 1 \tag{41}$$

we find from eq. (39)

$$\int d^3\vec{x}\ j_0(t,\vec{x}) \ = \ \int d^3\vec{x}\ e\ \psi^\dagger(x)\psi(x) \ = \ e. \tag{42}$$

Also from classical mechanics we find from eq. (11)

$$\int d^3\vec{x}\ j_0(t,\vec{x}) = \int d^3\vec{x}\ e\ \delta^{(3)}(\vec{x}-\vec{X}(t))\ =\ e. \tag{43}$$

Eqs. (42) and (43) implies that the classical mechanics and quantum mechanics give the same value of the electric charge e of the electron even if the charge density distributions in classical mechanics and in quantum mechanics are different.

As we have seen above, since the charge of a point particle can not depend on the space coordinate $\vec{x}$, eqs. (39), (40), (41) and (42) suggest that quantum mechanics may provide a framework to determine the general form of the fundamental charge. For example, in electrodynamics eqs. (39), (40), (41) and (42) suggest that the fundamental electric charge e of the electron can not be time dependent but it has to be constant.

Let us apply this procedure to Yang-Mills theory to find the general form of eight time dependent fundamental color charges $q^a(t)$ of a quark where $a=1,2,...8$ are color indices.

IV. COLOR CHARGES OF A QUARK IN YANG-MILLS THEORY ARE TIME DEPENDENT

In Yang-Mills theory the lagrangian density of the quark in the presence of non-abelian Yang-Mills potential $A^{\mu a}(x)$ is given by [1, 2]

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}(x)F^{\mu\nu a}(x) + \bar{\psi}_i(x)[\delta_{ij}\gamma_\mu\partial^\mu - m\delta_{ij} - gT^a_{ij}\gamma_\mu A^{\mu a}(x)]\psi_j(x) \tag{44}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \tag{45}$$

with $a=1,2,...8$, and $i,j=1,2,3$ in SU(3).

From eq. (44) one finds that the Dirac equation of the quark in the presence of non-abelian Yang-Mills potential $A^{\mu a}(x)$ is given by

$$[\delta_{ij}\gamma_\mu\partial^\mu - m\delta_{ij} - gT^a_{ij}\gamma_\mu A^{\mu a}(x)]\psi_j(x) = 0. \tag{46}$$

Similarly from eq. (44) one finds that the Yang-Mills color current density $j^{\mu a}(x)$ generated by the color charges of the quark in Yang-Mills theory is given by [1, 2]

$$j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T^a_{ij}\psi_j(x) \tag{47}$$

which satisfies the equation

$$D_\mu[A]j^{\mu a}(x) = 0 \tag{48}$$

where

$$D_\mu^{ab}[A] = \delta^{ab}\partial_\mu + gf^{acb}A_\mu^c(x). \tag{49}$$

Since eq. (48) is not a continuity equation ($\partial_\mu j^{\mu a}(x) \neq 0$) one finds from eqs. (48) and (47) that the color charge of the quark in Yang-Mills theory is not constant. As we have seen above, since the color charge can be obtained from the zero component of a corresponding color current density after integrating over the entire (physically) allowed volume $V = \int d^3\vec{x}$, one finds that the color charge of the quark can not depend on the space coordinate $\vec{x}$.

Since the color charge of the quark is not constant and it can not depend on space coordinate $\vec{x}$, one finds that the color charge of the quark has to be time dependent. Since the color current density $j^{\mu a}(x)$ of a quark in eq. (47) has eight color components $a = 1, 2, ...8$ we find that there are eight time dependent fundamental color charges of a quark.

We denote eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) by $q^a(t)$ where $a = 1, 2, ...8$ are color indices. These time dependent fundamental color charges $q^a(t)$ of the quark are independent of quark flavor, *i.e.*, a color charge $q^a(t)$ of the u (up) quark is same as that of d (down), S (strange), c (charm), B (bottom) or t (top) quark.

V. RELATION BETWEEN FUNDAMENTAL COUPLING CONSTANT AND FUNDAMENTAL CHARGE(S) IN MAXWELL THEORY AND IN YANG-MILLS THEORY

Let us now ask a fundamental question in nature. Since strong force or color force is produced from the color charges in nature "what is the relation between fundamental strong coupling constant α_s and the fundamental color charges $q^a(t)$ in the classical Yang-Mills theory" ?

One expects such a question because we know that a fundamental coupling constant is the measure of the strength of a fundamental force in the nature produced by the fundamental charges.

For example the fundamental electromagnetic force in nature is produced from the fundamental electric charge $\pm e$. The strength of this fundamental electromagnetic force in the nature is characterized by the typical value of the fundamental electromagnetic coupling constant (or fine structure constant)

$$\alpha = \frac{(\pm e)^2}{4\pi} \sim \frac{1}{137} \tag{50}$$

which is related to the fundamental electric charge $\pm e$ in Maxwell theory.

Similarly, the fundamental strong force or color force inside a hadron is produced from the fundamental color charges $q^a(t)$ in nature. Since the strength of the fundamental strong force or color force is characterized by the value of the fundamental strong coupling constant α_s one anticipates that there is a relation between the fundamental strong coupling constant α_s and the fundamental color charges $q^a(t)$ in Yang-Mills theory where $a = 1, 2, ...8$.

It is useful to denote $N^2 - 1$ color charges $q^a(t)$ of a fermion in SU(N) Yang-Mills theory as components of a single color charge vector $\vec{q}(t)$ in $N^2 - 1$ dimensional color space (group space). For example, in SU(3) Yang-Mills theory the eight time dependent fundamental color charges $q^a(t)$ of a quark can be denoted by a single vector $\vec{q}(t)$ in the eight dimensional color space (group space in SU(3)) where $a = 1, 2, ...8$. We call $\vec{q}(t)$ as eight-vector in color space, similar to three-vectors $\vec{x}(t)$ and $\vec{v}(t)$ in coordinate space. Note that although the three-vector $\vec{x}(t)$ in coordinate space is not rotationally invariant, its magnitude $|\vec{x}(t)|$ is rotationally invariant. Similarly, one finds that even if the eight-vector $\vec{q}(t)$ in color space is not gauge invariant, its magnitude $|\vec{q}(t)|$ is gauge invariant where

$$\vec{q}^2(t) = |\vec{q}(t)|^2 = \sum_{a=1}^{8} q^a(t)q^a(t). \tag{51}$$

In Maxwell theory the electromagnetic coupling constant (or fine structure constant) is related to the magnitude e^2 of the fundamental electric charge '$-e$' of the electron via the equation

$$\alpha = \frac{(-e)^2}{4\pi}. \tag{52}$$

Similarly, since the strong coupling constant α_s is gauge invariant, one finds by extending eq. (52) to Yang-Mills theory that the strong coupling constant α_s is related to the magnitude $\vec{q}^2(t)$ of the time dependent fundamental color charge '$\vec{q}(t)$' of the quark via the equation

$$\alpha_s = \frac{\vec{q}^2(t)}{4\pi}. \tag{53}$$

Note that even if the fundamental color charge $\vec{q}(t)$ of the quark is time dependent, the gauge invariant $\vec{q}^2(t) = \sum_{a=1}^{8} q^a(t)q^a(t)$ can be time independent.

In SU(2) Yang-Mills theory the color charge vector $\vec{q}(t)$ has three components $q_i(t)$ where $i = 1, 2, 3$.

Since the color charge vector $\vec{q}(t)$ has eight components in SU(3) Yang-Mills theory we find from eq. (53)

$$\alpha_s = \frac{q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4^2(t) + q_5^2(t) + q_6^2(t) + q_7^2(t) + q_8^2(t)}{4\pi} \tag{54}$$

in SU(3) Yang-Mills theory.

Similarly, since the color charge vector $\vec{q}(t)$ has three components in SU(2) Yang-Mills theory we find from eq. (53)

$$\alpha_s = \frac{q_1^2(t) + q_2^2(t) + q_3^2(t)}{4\pi} \tag{55}$$

in SU(2) Yang-Mills theory.

Since the universal coupling g in Yang-Mills theory and the strong coupling constant α_s are related by

$$\alpha_s = \frac{g^2}{4\pi} \tag{56}$$

we find from eq. (53) that

$$g^2 = \vec{q}^2(t). \tag{57}$$

In SU(3) Yang-Mills theory we find from eq. (57)

$$g^2 = q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4^2(t) + q_5^2(t) + q_6^2(t) + q_7^2(t) + q_8^2(t) \tag{58}$$

and in SU(2) Yang-Mills theory we find from eq. (57)

$$g^2 = q_1^2(t) + q_2^2(t) + q_3^2(t). \tag{59}$$

VI. COLOR CURRENT DENSITY AND COLOR CHARGE OF QUARK AND GENERAL FORM OF YANG-MILLS POTENTIAL

Extending eq. (11) to time dependent charge $q^a(t)$ we find that the abelian-like non-abelian current density $\mathcal{J}^{\mu a}(t, \vec{x})$ produced from time dependent charge $q^a(t)$ in classical

chromodynamics in an inertial frame K is given by

$$\mathcal{J}_0^a(t,\vec{x}) = q^a(t)\ \delta^{(3)}(\vec{x}-\vec{X}(t))$$
$$\vec{\mathcal{J}}^a(t,\vec{x}) = q^a(t)\ \vec{v}(t)\ \delta^{(3)}(\vec{x}-\vec{X}(t)) \tag{60}$$

which satisfies the equation

$$\partial_\mu \mathcal{J}^{\mu a}(t,\vec{x}) = \frac{dq^a(t)}{dt}\ \delta^{(3)}(\vec{x}-\vec{X}(t)). \tag{61}$$

In a covariant formulation we find from eq. (60)

$$\mathcal{J}_\mu^a(x) = \int d\tau\ q^a(\tau)\ u_\mu(\tau)\ \delta^{(4)}(x - X(\tau)). \tag{62}$$

Similarly in covariant formulation we find from eq. (61)

$$\partial_\mu \mathcal{J}^{\mu a}(x) = \int d\tau \frac{dq^a(\tau)}{d\tau}\ \delta^{(4)}(x-X(\tau)) \neq 0. \tag{63}$$

Since eq. (63) is not a continuity equation one finds that the charge $q^a(t)$ is not constant.

The solution of the inhomogeneous wave equation

$$\partial_\nu \partial^\nu \mathcal{A}_\mu^a(x) = \mathcal{J}_\mu^a(x). \tag{64}$$

is given by

$$\mathcal{A}_\mu^a(x) = \int d^4x' D_r(x-x')\mathcal{J}_\mu^a(x') \tag{65}$$

where $D_r(x-x')$ is the retarded Greens function [7]. When $\mathcal{A}_\mu^a(x)$ satisfies the Lorentz gauge condition

$$\partial^\mu \mathcal{A}_\mu^a(x) = 0, \tag{66}$$

then eq. (64) reduces to the inhomogeneous Maxwell-like equation

$$\partial^\mu \mathcal{F}_{\mu\nu}^a(x) = \mathcal{J}_\nu^a(x) \tag{67}$$

where

$$\mathcal{F}_{\mu\nu}^a(x) = \partial_\mu \mathcal{A}_\nu^a(x) - \partial_\nu \mathcal{A}_\mu^a(x). \tag{68}$$

Hence we find that if $\mathcal{A}_\mu^a(x)$ obtained from eq. (64) satisfies Lorentz gauge condition eq. (66) then it satisfies Maxwell-like eq. (67) where $\mathcal{F}_{\mu\nu}^a(x)$ is given by eq. (68).

Using eq. (62) in (65) we find that

$$\mathcal{A}^a_\mu(x) = q^a(\tau_0)\frac{u_\mu(\tau_0)}{u(\tau_0)\cdot(x-X(\tau_0))} \tag{69}$$

where τ_0 is determined from the solution of the retarded equation

$$x_0 - X_0(\tau_0) = |\vec{x} - \vec{X}(\tau_0)|. \tag{70}$$

From eqs. (69) and (70) we find

$$\partial^\nu \mathcal{A}^a_\mu(x) = q^a(\tau_0)\frac{(x-X(\tau_0))^\nu \dot{u}_\mu(\tau_0)}{[u(\tau_0)\cdot(x-X(\tau_0))]^2} - q^a(\tau_0)\frac{u_\mu(\tau_0)}{[u(\tau_0)\cdot(x-X(\tau_0))]^2}$$
$$[\frac{[\dot{u}(\tau_0)\cdot(x-X(\tau_0))-1](x-X(\tau_0))^\nu}{u(\tau_0)\cdot(x-X(\tau_0))} + u^\nu(\tau_0)] + [\partial^\nu q^a(\tau_0)]\frac{u_\mu(\tau_0)}{u(\tau_0)\cdot(x-X(\tau_0))} \tag{71}$$

which gives

$$\partial_\nu\partial^\nu \mathcal{A}^a_\mu(x) = 0. \tag{72}$$

Since $x^\mu \neq X^\mu(\tau_0)$ is the observation point of the potential (or any gauge invariant obtained from the potential where $X^\mu(\tau_0)$ is the time-space position of the charge), we find that eq. (72) is expected because eq. (69) is obtained from eq. (64) where $\mathcal{J}^a_\mu(x)$ is given by eq. (62).

From eq. (71) we find

$$\partial^\mu \mathcal{A}^a_\mu(x) = \frac{\dot{q}^a(\tau_0)}{u(\tau_0)\cdot(x-X(\tau_0))}, \qquad \dot{q}^a(\tau_0) = \frac{dq^a(\tau)}{d\tau}|_{\tau=\tau_0}. \tag{73}$$

Since a quark has non-zero mass it can not travel exactly at speed of light $v = c$. When the quark is in uniform motion and is at its highest speed (which is arbitrarily close to speed of light $v \sim c$) we find from eq. (73) that the Lorentz gauge condition is (approximately) satisfied, *i.e.*,

$$\partial^\mu \mathcal{A}^a_\mu(x) \sim 0, \qquad \text{when} \qquad v \sim c. \tag{74}$$

From eq. (69) we find that the abelian-like non-abelian potential produced by the quark in uniform motion at its highest speed (which is arbitrarily close to the speed of light ($v \sim c$)) is given by

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0)\frac{\beta^\mu_{\sim c}}{\beta_{\sim c}\cdot(x-X(\tau_0))}, \qquad \beta^\mu_{\sim c} = (1,\ \vec{\beta}_{\sim c}), \qquad \vec{\beta}^2_{\sim c} = \frac{v^2}{c^2} \sim 1. \tag{75}$$

From eqs. (72), (74) and (68) we find that $\mathcal{A}^{\mu a}(x)$ in eq. (75) satisfies the equation

$$\partial^\nu \mathcal{F}^a_{\nu\mu}(x) \sim 0. \tag{76}$$

Since $x^\mu \neq X^\mu(\tau_0)$ is the observation point of the potential (or any gauge invariant obtained from the potential where $X^\mu(\tau_0)$ is the time-space position of the charge), we find that eq. (76) (approximately) satisfies Maxwell-like equation (67) where $\mathcal{J}^a_\mu(x)$ is given by eq. (62).

From eqs. (69) and (70) we find

$$\begin{aligned}
&\partial_\mu \mathcal{A}^a_\nu(x) - \partial_\nu \mathcal{A}^a_\mu(x) = q^a(\tau_0)\frac{(x-X(\tau_0))_\mu \dot{u}_\nu(\tau_0) - (x-X(\tau_0))_\nu \dot{u}_\mu(\tau_0)}{[u(\tau_0)\cdot(x-X(\tau_0))]^2} \\
&-q^a(\tau_0)\ [\frac{[\dot{u}(\tau_0)\cdot(x-X(\tau_0))-1][(x-X(\tau_0))_\mu u_\nu(\tau_0) - (x-X(\tau_0))_\nu u_\mu(\tau_0)]}{[u(\tau_0)\cdot(x-X(\tau_0))]^3}] \\
&+[\dot{q}^a(\tau_0)]\frac{(x-X(\tau_0))_\mu u_\nu(\tau_0) - (x-X(\tau_0))_\nu u_\mu(\tau_0)}{[u(\tau_0)\cdot(x-X(\tau_0))]^2}.
\end{aligned} \tag{77}$$

When the quark in uniform motion is at its highest speed (arbitrarily close to speed of light $v \sim c$) we find from eq. (77)

$$\mathcal{F}^a_{\mu\nu}(x) \sim 0, \qquad \text{when} \qquad v \sim c \tag{78}$$

where $\mathcal{F}^a_{\mu\nu}(x)$ is given by eq. (68).

From eqs. (75), (74), (76) and (78) we find that

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0)\frac{\beta^\mu_{\sim c}}{\beta_{\sim c}\cdot(x-X(\tau_0))} \tag{79}$$

satisfies

$$\partial^\mu \mathcal{A}^a_\mu(x) \sim 0, \tag{80}$$

$$\partial^\mu \mathcal{F}^a_{\mu\nu}(x) \sim 0 \tag{81}$$

and

$$\mathcal{F}^a_{\mu\nu}(x) \sim 0 \tag{82}$$

where $\mathcal{F}^a_{\mu\nu}(x)$ is given by eq. (68).

From eqs. (79), (82) and (68) we find that eq. (79) can be written as

$$\mathcal{A}^{\mu a} = q^a(\tau_0)\frac{\beta^\mu_{\sim c}}{\beta_{\sim c}\cdot(x-X(\tau_0))} \sim \partial^\mu \omega^a(x), \qquad \omega^a(x) = \int dl_c\ \frac{q^a(\tau_0)}{l_c}, \qquad l_c = \beta_{\sim c}\cdot(x-X(\tau_0)). \tag{83}$$

Hence we find that the general form of abelian-like non-abelian (approximate) pure gauge potential produced by the quark with time dependent color charge $q^a(t)$ is given by eq. (83).

We call eq. (83) as abelian-like non-abelian (approximate) pure gauge potential because a quark has non-zero mass (even if very small) and hence it can not travel exactly at speed of light $\beta_c^\mu = (1,\ \vec{\beta}_c)$ or $\vec{\beta}^2_{\sim c} = \frac{v^2}{c^2} = 1$ to produce the exact abelian-like non-abelian pure gauge potential

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0)\frac{\beta_c^\mu}{\beta_c \cdot (x - X(\tau_0))} = \partial^\mu \omega^a(x), \qquad \omega^a(x) = \int dl_c\ \frac{q^a(\tau_0)}{l_c}, \qquad l_c = \beta_c \cdot (x - X(\tau_0)). \tag{84}$$

Note that in Maxwell theory the abelian pure gauge potential is given by

$$A^\mu(x) = -\frac{1}{ie}[\partial^\mu U(x)]U^{-1}(x) = \partial^\mu \omega(x), \qquad U(x) = e^{-ie\omega(x)} \tag{85}$$

where (see eq. (33))

$$\omega(x) \propto e. \tag{86}$$

In Yang-Mills theory the SU(3) non-abelian pure gauge potential $A^{\mu a}(x)$ in

$$T^a A^{\mu a}(x) = \frac{1}{ig}[\partial^\mu U(x)]U^{-1}(x), \qquad U(x) = e^{igT^a\omega^a(x)} \tag{87}$$

contains infinite number of terms up to infinite powers of g and/or up to infinite powers of $\omega^a(x)$ where

$$\omega^a(x) \propto g. \tag{88}$$

The first term in the expansion in eq. (87) is the abelian-like non-abelian pure gauge potential given by

$$\mathcal{A}^{\mu a}(x) = \partial^\mu \omega^a(x) \tag{89}$$

where

$$\omega^a(x) \propto q^a(\tau_0) \propto g, \tag{90}$$

see eqs. (83) and (57).

From eqs. (83) and (87) we find that the SU(3) (approximate) pure gauge potential in Yang-Mills theory is given by

$$A^{\mu a}(x) = \frac{\beta^\mu_{\sim c}}{\beta_{\sim c}\cdot(x-X(\tau_0))} q^b(\tau_0)[\frac{e^{g\int dl_c \frac{Q(\tau_0)}{l_c}}-1}{g\int dl_c \frac{Q(\tau_0)}{l_c}}]_{ab} \sim [\partial_\mu \omega^b(x)]\,[\frac{e^{gM(x)}-1}{gM(x)}]_{ab} \tag{91}$$

where $\int dl_c$ is an indefinite integration and

$$Q^{ab}(\tau_0) = f^{abc}q^c(\tau_0), \qquad l_c = \beta_{\sim c}\cdot(x-X(\tau_0)), \qquad M_{ab}(x) = f^{abc}\omega^c(x). \tag{92}$$

In eqs. (91) and (92) the repeated indices $b, c = 1, 2, ...8$ are summed and τ_0 is determined from the solution of the retarded equation

$$x_0 - X_0(\tau_0) = |\vec{x} - \vec{X}(\tau_0)|. \tag{93}$$

From eq. (91) we find that the general form of the Yang-Mills potential $A^{\mu a}(x)$ produced by the quark moving with arbitrary four-velocity $u^\mu(\tau) = \frac{dX^\mu(\tau)}{d\tau}$ is given by

$$A^{\mu a}(x) = \frac{u^\mu(\tau_0)}{u(\tau_0)\cdot(x-X(\tau_0))} q^b(\tau_0)[\frac{e^{g\int dl \frac{Q(\tau_0)}{l}}-1}{g\int dl \frac{Q(\tau_0)}{l}}]_{ab} \tag{94}$$

where $\int dl$ is an indefinite integration and

$$Q^{ab}(\tau_0) = f^{abc}q^c(\tau_0), \qquad l = u(\tau_0)\cdot(x-X(\tau_0)). \tag{95}$$

In eqs. (94) and (95) the repeated indices $b, c = 1, 2, ...8$ are summed and τ_0 is determined from the solution of the retarded equation

$$x_0 - X_0(\tau_0) = |\vec{x} - \vec{X}(\tau_0)|. \tag{96}$$

From eqs. (94), (44), (47), (49) and (45) we find that the non-abelian Yang-Mills color current density $j^{\mu a}(x)$ of the quark which satisfies the equation

$$\begin{aligned} &j^{\mu a}(x) = D_\nu[A]F^{\nu\mu a}(x) = \\ &\partial^\nu[\partial_\nu A^a_\mu(x) - \partial_\mu A^a_\nu(x) + gf^{abc}A^b_\nu(x)A^c_\mu(x)] + gf^{abc}A^{\nu c}[\partial_\nu A^b_\mu(x) - \partial_\mu A^b_\nu(x) + gf^{bhd}A^h_\nu(x)A^d_\mu(x)] \end{aligned} \tag{97}$$

is non-linear function of g or non-linear function of $q^a(\tau)$. Since the abelian-like non-abelian color current density $\mathcal{J}^a_\mu(x)$ of the quark in eq. (60) is linearly proportional to g (see eq.

(90)) and the non-abelian Yang-Mills color current density $j^a_\mu(x)$ of the quark in eq. (97) is non-linear function of g, we find that $\int d^3\vec{x}\ \mathcal{J}^a_0(t,\vec{x})$ in eq. (60) is different from $\int d^3\vec{x}\ j^a_0(t,\vec{x})$ in eq. (97) in Yang-Mills theory. Note that they are same for electron in Maxwell theory because $A^\mu(x)$ and $F^{\mu\nu}(x)$ are linearly proportional to the charge e of the electron in Maxwell theory.

From eqs. (47) and (97) we find that the non-abelian Yang-Mills color current density

$$j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T^a_{ij}\psi_j(x) \tag{98}$$

of the quark is non-linear function of g. Note that in Maxwell theory $e\bar{\psi}(x)\gamma^\mu\psi(x)$ is linearly proportional to e.

Hence we find that, unlike Maxwell theory where $\int d^3\vec{x}\ \psi^\dagger(t,\vec{x})\psi(t,\vec{x}) = 1$ is independent of e, we find from eqs. (97) and (98) that in Yang-Mills theory $\int d^3\vec{x}\ \psi^\dagger_i(t,\vec{x})T^a_{ij}\psi_j(t,\vec{x})$ is non-linear function of g even if the wave function of the quark is normalized, *i.e.*, even if

$$\sum_{i=1}^{3}\int d^3\vec{x}\ \vec{x}\ \psi^\dagger_i(t,\vec{x})\psi_i(t,\vec{x}) = 1. \tag{99}$$

VII. FERMION COLOR CURRENT DENSITY, FERMION WAVE FUNCTION AND PAULI MATRICES IN YANG-MILLS THEORY IN SU(2)

In SU(2) Yang-Mills theory the Yang-Mills color current density is given by

$$j^i_\mu(x) = g\bar{\psi}_k(x)\gamma_\mu\tau^i_{kn}\psi_n(x); \qquad i=1,2,3; \qquad k,n=1,2; \tag{100}$$

where Pauli matrices are given by

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & ,1 \\ 1 & ,0 \end{pmatrix} \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & ,-i \\ i & ,0 \end{pmatrix} \quad \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & ,0 \\ 0 & ,-1 \end{pmatrix}. \tag{101}$$

If there is only one fermion in the entire (physically) allowed volume $V = \int d^3\vec{x}$ then the total probability of finding that fermion in the entire volume is 1. This implies that the normalized wave function $\psi_k(x)$ of the fermion, where $k = 1,\ 2$ are the color indices of the fermion wave function in SU(2) Yang-Mills theory, satisfies the normalization condition

$$\int d^3\vec{x}\ [\psi^\dagger_1(x)\psi_1(x) + \psi^\dagger_2(x)\psi_2(x)] = 1. \tag{102}$$

From eqs. (100) and (101) we find

$$\int d^3\vec{x} j_0^1(x) = \frac{g}{2}\int d^3\vec{x}\psi_1^\dagger(x)\psi_2(x) + \frac{g}{2}\int d^3\vec{x}\psi_2^\dagger(x)\psi_1(x),$$
$$\int d^3\vec{x} j_0^2(x) = -\frac{ig}{2}\int d^3\vec{x}\psi_1^\dagger(x)\psi_2(x) + \frac{ig}{2}\int d^3\vec{x}\psi_2^\dagger(x)\psi_1(x),$$
$$\int d^3\vec{x} j_0^3(x) = \frac{g}{2}\int d^3\vec{x}\psi_1^\dagger(x)\psi_1(x) - \frac{g}{2}\int d^3\vec{x}\psi_2^\dagger(x)\psi_2(x). \tag{103}$$

As we have seen above, since $\int d^3\vec{x}\ \psi_i^\dagger(t,\vec{x})\tau_{ij}^a\psi_j(t,\vec{x})$ in Yang-Mills theory is non-linear function of g we write

$$d_{11}(t,g) = \int d^3\vec{x}\psi_1^\dagger(x)\psi_1(x), \qquad d_{22}(t,g) = \int d^3\vec{x}\psi_2^\dagger(x)\psi_2(x), \qquad d_{12}(t,g) = \int d^3\vec{x}\psi_1^\dagger(x)\psi_2(x) \tag{104}$$

where $d_{11}(t,g)$, $d_{22}(t,g)$ are t and g dependent real positive functions and $d_{12}(t,g)$ is t and g dependent complex function.

From eqs. (103) and (104) we find

$$\int d^3\vec{x} j_0^1(x) = \frac{g}{2}\times[d_{12}(t,g) + d_{12}^*(t,g)]$$
$$\int d^3\vec{x} j_0^2(x) = -\frac{ig}{2}\times[d_{12}(t,g) - d_{12}^*(t,g)]$$
$$\int d^3\vec{x} j_0^3(x) = \frac{g}{2}\times[d_{11}(t,g) - d_{22}(t,g)]. \tag{105}$$

From the normalization condition, see eq. (102), we find

$$d_{11}(t,g) + d_{22}(t,g) = 1 \tag{106}$$

where we have used eq. (104).

Since $d_{11}(t,g)$ and $d_{22}(t,g)$ are t and g dependent real positive functions (see eq. (104)) we can write eq. (106) as

$$d_{11}(t,g) = \cos^2\Theta(t,g), \qquad d_{22}(t,g) = \sin^2\Theta(t,g) \tag{107}$$

where

$$0 \le \Theta(t,g) \le 2\pi. \tag{108}$$

Using eq. (107) in (105) we find

$$\int d^3\vec{x} j_0^1(x) = \frac{g}{2}\times[d_{12}(t,g) + d_{12}^*(t,g)]$$
$$\int d^3\vec{x} j_0^2(x) = -\frac{ig}{2}\times[d_{12}(t,g) - d_{12}^*(t,g)]$$
$$\int d^3\vec{x} j_0^3(x) = \frac{g}{2}\times\cos[2\Theta(t,g)]. \tag{109}$$

VIII. GENERAL FORM OF FUNDAMENTAL COLOR CHARGE OF A FERMION IN YANG-MILLS THEORY IN SU(2)

Since the fundamental color charge vector $\vec{q}(t)$ is linearly proportional to g (see eqs. (57) and (90)) we find from eq. (109) that the color charge $q^i(t)$ of a fermion in Yang-Mills theory in SU(2) takes the form

$$
\begin{aligned}
q_1(t) &= \frac{g}{2} \times [d_{12}(t) + d^*_{12}(t)] \\
q_2(t) &= -\frac{ig}{2} \times [d_{12}(t) - d^*_{12}(t)] \\
q_3(t) &= \frac{g}{2} \times \cos[2\Theta(t)]
\end{aligned}
\tag{110}
$$

where the complex function $d_{12}(t)$ and the real phase factor $\Theta(t)$ depend on time t but are independent of g. Since $d_{12}(t)$ is a complex function we can write eq. (110) as

$$
\begin{aligned}
q_1(t) &= g \times |d_{12}(t)| \times \cos\phi(t) \\
q_2(t) &= g \times |d_{12}(t)| \times \sin\phi(t) \\
q_3(t) &= \frac{g}{2} \times \cos[2\Theta(t)]
\end{aligned}
\tag{111}
$$

where the principal argument

$$
\mathrm{Arg}\,(d_{12}(t)) = \phi(t) = \tan^{-1}\left[\frac{\mathrm{Im}[d_{12}(t)]}{\mathrm{Re}[d_{12}(t)]}\right].
\tag{112}
$$

of the complex function $d_{12}(t)$ lying in the range

$$
-\pi \le \phi(t) \le \pi.
\tag{113}
$$

It is important to remember that the real phase factors $\Theta(t)$ and $\phi(t)$ in eq. (111) are not independent of time t. This is because if the real phase factors $\Theta(t)$ and $\phi(t)$ are independent of time t then the non-abelian Yang-Mills potential $A^{\mu a}(x)$ in eq. (94) becomes Maxwell-like potential $A^\mu(x)$.

From eqs. (59) and (111) we find

$$
|d_{12}(t)|^2 + \frac{1}{4}\cos^2[2\Theta(t)] = 1.
\tag{114}
$$

From eq. (108) we find that the maximum allowed range of $\Theta(t)$ is

$$
0 \le \Theta(t) \le 2\pi.
\tag{115}
$$

From eqs. (114) and (115) we find

$$\frac{\sqrt{3}}{2} \le |d_{12}(t)| \le 1. \tag{116}$$

We write

$$\frac{1}{4}\cos^2[2\Theta(t)] = \cos^2\theta(t) \tag{117}$$

where from eqs. (115) and (117) we find

$$0 \le \cos^2\theta(t) \le \frac{1}{4}. \tag{118}$$

Since $|d_{12}(t)|$ is positive we find from eqs. (114), (116), (117) and (118) that we can write

$$|d_{12}(t)| = \sin\theta(t) \tag{119}$$

where

$$\frac{\pi}{3} \le \theta(t) \le \frac{2\pi}{3}. \tag{120}$$

Hence from eqs. (117), (119), (120), (113) and (111) we find that three time dependent fundamental color charges of a fermion in Yang-Mills theory in SU(2) are given by

$$q_1(t) = g \times \sin\theta(t) \times \cos\phi(t),$$
$$q_2(t) = g \times \sin\theta(t) \times \sin\phi(t),$$
$$q_3(t) = g \times \cos\theta(t) \tag{121}$$

which reproduces eq. (5) where

$$\frac{\pi}{3} \le \theta(t) \le \frac{2\pi}{3}, \qquad\qquad -\pi \le \phi(t) \le \pi. \tag{122}$$

Hence we find that the general form of three time dependent fundamental color charges of a fermion in Yang-Mills theory in SU(2) is given by eq. (121) where the ranges of $\theta(t)$ and $\phi(t)$ are given by eq. (122). Note that the real phase factors $\theta(t)$ and $\phi(t)$ in eq. (121) can not be independent of time t because if they are independent of t then the Yang-Mills potential $A^{\mu a}(x)$ in eq. (94) will reduce to Maxwell-like potential $A^{\mu}(x)$.

IX. YANG-MILLS COLOR CURRENT DENSITY, QUARK WAVE FUNCTION AND GELL-MANN MATRICES IN YANG-MILLS THEORY IN SU(3)

Eight Gell-Mann matrices in SU(3) Yang-Mills theory are given by

$$T_1=\frac{1}{2}\begin{pmatrix}0, & 1, & 0\\ 1, & 0, & 0\\ 0, & 0, & 0\end{pmatrix},\quad T_2=\frac{1}{2}\begin{pmatrix}0, & -i, & 0\\ i, & 0, & 0\\ 0, & 0, & 0\end{pmatrix},\quad T_3=\frac{1}{2}\begin{pmatrix}1, & 0, & 0\\ 0, & -1, & 0\\ 0, & 0, & 0\end{pmatrix},\quad T_4=\frac{1}{2}\begin{pmatrix}0, & 0, & 1\\ 0, & 0, & 0\\ 1, & 0, & 0\end{pmatrix},$$
$$T_5=\frac{1}{2}\begin{pmatrix}0, & 0, & -i\\ 0, & 0, & 0\\ i, & 0, & 0\end{pmatrix},\quad T_6=\frac{1}{2}\begin{pmatrix}0, & 0, & 0\\ 0, & 0, & 1\\ 0, & 1, & 0\end{pmatrix},\quad T_7=\frac{1}{2}\begin{pmatrix}0, & 0, & 0\\ 0, & 0, & -i\\ 0, & i, & 0\end{pmatrix},\quad T_8=\frac{1}{2\sqrt{3}}\begin{pmatrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & -2\end{pmatrix}. \tag{123}$$

In Yang-Mills theory in SU(3) the Yang-Mills color current density of the quark is given by

$$j^{\mu a}(x)=g\bar{\psi}_i(x)\gamma^\mu T^a_{ij}\psi_j(x);\qquad a=1,2,...8;\qquad i,j=1,2,3. \tag{124}$$

The normalized wave function $\psi_i(x)$ of the quark obey the equation

$$\int d^3\vec{x}\ \psi_1^\dagger(x)\psi_1(x)+\int d^3\vec{x}\ \psi_2^\dagger(x)\psi_2(x)+\int d^3\vec{x}\ \psi_3^\dagger(x)\psi_3(x)=1. \tag{125}$$

From eqs. (124) and (123) we find

$$\begin{aligned}
\int d^3\vec{x}j_0^1(x)&=\frac{g}{2}[h_{12}(t,g)+h_{12}^*(t,g)],\\
\int d^3\vec{x}j_0^2(x)&=-\frac{ig}{2}[h_{12}(t,g)-h_{12}^*(t,g)],\\
\int d^3\vec{x}j_0^3(x)&=\frac{g}{2}[h_{11}(t,g)-h_{22}(t,g)],\\
\int d^3\vec{x}j_0^4(x)&=\frac{g}{2}[h_{13}(t,g)+h_{13}^*(t,g)],\\
\int d^3\vec{x}j_0^5(x)&=-\frac{ig}{2}[h_{13}(t,g)-h_{13}^*(t,g)],\\
\int d^3\vec{x}j_0^6(x)&=\frac{g}{2}[h_{23}(t,g)+h_{23}^*(t,g)],\\
\int d^3\vec{x}j_0^7(x)&=-\frac{ig}{2}[h_{23}(t,g)-h_{23}^*(t,g)],\\
\int d^3\vec{x}j_0^8(x)&=\frac{g}{2\sqrt{3}}[h_{11}(t,g)+h_{22}(t,g)-2h_{33}(t,g)]
\end{aligned} \tag{126}$$

where

$$h_{11}(t,g)=\int d^3\vec{x}\psi_1^\dagger(x)\psi_1(x),\qquad h_{22}(t,g)=\int d^3\vec{x}\psi_2^\dagger(x)\psi_2(x),\qquad h_{33}(t,g)=\int d^3\vec{x}\psi_3^\dagger(x)\psi_3(x),$$

$$h_{12}(t,g) = \int d^3\vec{x}\psi_1^\dagger(x)\psi_2(x), \qquad h_{13}(t,g) = \int d^3\vec{x}\psi_1^\dagger(x)\psi_3(x), \qquad h_{23}(t,g) = \int d^3\vec{x}\psi_2^\dagger(x)\psi_3(x) \tag{127}$$

and from eq. (125) we find

$$h_{11}(t,g) + h_{22}(t,g) + h_{33}(t,g) = 1. \tag{128}$$

From eq. (127) we find that $h_{11}(t,g)$, $h_{22}(t,g)$, $h_{33}(t,g)$ are t and g dependent real positive functions and $h_{12}(t,g)$, $h_{13}(t,g)$ and $h_{23}(t,g)$ are t and g dependent complex functions.

From now onwards we can proceed exactly in the way similar to SU(2).

Since $h_{11}(t,g)$, $h_{22}(t,g)$ and $h_{33}(t,g)$ are t and g dependent real positive functions (see eq. (127)) we can write eq. (128) as

$$h_{11}(t,g) = \sin^2\Theta(t,g) \times \cos^2\Phi(t,g), \qquad h_{22}(t,g) = \sin^2\Theta(t,g) \times \sin^2\Phi(t,g), \qquad h_{33}(t,g) = \cos^2\Theta(t,g) \tag{129}$$

where

$$0 \le \Theta(t,g) \le \pi, \qquad\qquad 0 \le \Phi(t,g) \le 2\pi. \tag{130}$$

Using eq. (129) in (126) we find

$$\begin{aligned}
\int d^3\vec{x} j_0^1(x) &= \frac{g}{2} \times [h_{12}(t,g) + h_{12}^*(t,g)], \\
\int d^3\vec{x} j_0^2(x) &= -\frac{ig}{2} \times [h_{12}(t,g) - h_{12}^*(t,g)], \\
\int d^3\vec{x} j_0^3(x) &= \frac{g}{2} \times \sin^2\Theta(t,g) \times \cos[2\Phi(t,g)] \\
\int d^3\vec{x} j_0^4(x) &= \frac{g}{2} \times [h_{13}(t,g) + h_{13}^*(t,g)], \\
\int d^3\vec{x} j_0^5(x) &= -\frac{ig}{2} \times [h_{13}(t,g) - h_{13}^*(t,g)], \\
\int d^3\vec{x} j_0^6(x) &= \frac{g}{2} \times [h_{23}(t,g) + h_{23}^*(t,g)], \\
\int d^3\vec{x} j_0^7(x) &= -\frac{ig}{2} \times [h_{23}(t,g) - h_{23}^*(t,g)], \\
\int d^3\vec{x} j_0^8(x) &= \frac{g}{2\sqrt{3}} \times [1 - 3\,\cos^2\Theta(t,g)].
\end{aligned} \tag{131}$$

X. GENERAL FORM OF FUNDAMENTAL COLOR CHARGE OF A QUARK IN YANG-MILLS THEORY IN SU(3)

Since the fundamental color charge vector $\vec{q}(t)$ is linearly proportional to g (see eqs. (57) and (90)) we find from eq. (131) that the color charge $q^a(t)$ of a quark in Yang-Mills theory in SU(3) takes the form

$$
\begin{aligned}
q_1(t) &= \frac{g}{2} \times [h_{12}(t) + h_{12}^*(t)],\\
q_2(t) &= -\frac{ig}{2} \times [h_{12}(t) - h_{12}^*(t)],\\
q_3(t) &= \frac{g}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)]\\
q_4(t) &= \frac{g}{2} \times [h_{13}(t) + h_{13}^*(t)],\\
q_5(t) &= -\frac{ig}{2} \times [h_{13}(t) - h_{13}^*(t)],\\
q_6(t) &= \frac{g}{2} \times [h_{23}(t) + h_{23}^*(t)],\\
q_7(t) &= -\frac{ig}{2} \times [h_{23}(t) - h_{23}^*(t)],\\
q_8(t) &= \frac{g}{2\sqrt{3}} \times [1 - 3\,\cos^2\Theta(t)].
\end{aligned}
\tag{132}
$$

where the complex functions $h_{12}(t)$, $h_{13}(t)$, $h_{23}(t)$ and the real phase factors $\Theta(t)$, $\Phi(t)$ depend on time t but are independent of g. Since $h_{12}(t)$, $h_{13}(t)$, $h_{23}(t)$ are complex functions we can write eq. (132) as

$$
\begin{aligned}
q_1(t) &= g \times |h_{12}(t)| \times \cos\phi_{12}(t)\\
q_2(t) &= g \times |h_{12}(t)| \times \sin\phi_{12}(t),\\
q_3(t) &= \frac{g}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)],\\
q_4(t) &= g \times |h_{13}(t)| \times \cos\phi_{13}(t),\\
q_5(t) &= g \times |h_{13}(t)| \times \sin\phi_{13}(t),\\
q_6(t) &= g \times |h_{23}(t)| \times \cos\phi_{23}(t),\\
q_7(t) &= g \times |h_{23}(t)| \times \sin\phi_{23}(t),\\
q_8(t) &= \frac{g}{2\sqrt{3}} \times [1 - 3\,\cos^2\Theta(t)]
\end{aligned}
\tag{133}
$$

where the principal arguments

$$\text{Arg}\,(h_{12}(t)) = \phi_{12}(t) = \tan^{-1}\left[\frac{\text{Im}[h_{12}(t)]}{\text{Re}[h_{12}(t)]}\right], \qquad \text{Arg}\,(h_{13}(t)) = \phi_{13}(t) = \tan^{-1}\left[\frac{\text{Im}[h_{13}(t)]}{\text{Re}[h_{13}(t)]}\right],$$

$$\mathrm{Arg}\,(h_{23}(t)) = \phi_{23}(t) = \tan^{-1}\left[\frac{\mathrm{Im}[h_{23}(t)]}{\mathrm{Re}[h_{23}(t)]}\right], \tag{134}$$

of the complex functions $h_{12}(t)$, $h_{13}(t)$ and $h_{23}(t)$ lying in the range

$$-\pi \le \phi_{12}(t), \quad \phi_{13}(t), \quad \phi_{23}(t) \le \pi. \tag{135}$$

From eqs. (58) and (133) we find

$$|h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 + \left[\frac{1}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)]\right]^2 \;+\; \left[\frac{1}{2\sqrt{3}} \times [1 - 3\,\cos^2\Theta(t)]\right]^2 = 1. \tag{136}$$

From eq. (130) we find that the maximum allowed range of $\Theta(t)$, $\Phi(t)$ are

$$0 \le \Theta(t) \le \pi, \qquad\qquad 0 \le \Phi(t) \le 2\pi. \tag{137}$$

In order to proceed further we need to find the maximum and minimum values of

$$\begin{aligned} H &= \left[\sin^2\Theta(t) \times \cos[2\Phi(t)]\right]^2 \;+\; \left[\frac{1}{\sqrt{3}} \times [1 - 3\,\cos^2\Theta(t)]\right]^2 \\ &= [\cos^2\Theta(t) - 1]^2 \times \cos^2[2\Phi(t)] + \frac{1}{3} + \;[3\,\cos^4\Theta(t) - 2\,\cos^2\Theta(t)] \end{aligned} \tag{138}$$

in the range of $\Theta(t)$, $\Phi(t)$ given by eq. (137). Taking the first derivative with respect to $\Phi(t)$ we find

$$\frac{dH}{d\Phi(t)} = -2\,\sin[4\Phi(t)] \times [\cos^2\Theta(t) - 1]^2. \tag{139}$$

Hence we find

$$\frac{dH}{d\Phi(t)} = 0 \tag{140}$$

when

$$\begin{aligned} &1)\ \cos^2\Theta(t) = 1, \qquad \text{for any} \qquad \Phi(t); \\ &2)\ \Phi(t) = 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ \frac{7\pi}{4},\ 2\pi, \qquad \text{for any} \qquad \Theta(t). \end{aligned} \tag{141}$$

Taking the first derivative with respect to $\Theta(t)$ we find

$$\frac{dH}{d\Theta(t)} = -2\,\sin[2\Theta(t)] \times \left[[\cos^2\Theta(t) - 1] \times \cos^2[2\Phi(t)] + 3\,\cos^2\Theta(t) - 1\right]. \tag{142}$$

Hence we find

$$\frac{dH}{d\Theta(t)} = 0 = \frac{dH}{d\Phi(t)} \tag{143}$$

when

$$\begin{aligned}
&1)\ \cos^2\Theta(t) = 1, && \text{for any} && \Phi(t);\\
&2)\ \Phi(t) = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi, && \text{and} && \cos^2\Theta(t) = \frac{1}{2}\\
&3)\ \Phi(t) = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}, && \text{and} && \cos^2\Theta(t) = \frac{1}{3}\\
&4)\ \Phi(t) = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi, && \text{and} && \Theta(t) = \frac{\pi}{2},\ \frac{3\pi}{2}\\
&5)\ \Phi(t) = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}, && \text{and} && \Theta(t) = \frac{\pi}{2},\ \frac{3\pi}{2}.
\end{aligned} \tag{144}$$

By taking the second derivative we find

$$\frac{d^2H}{d\Theta^2(t)} = -4\ \cos[2\Theta(t)] \times \Big[[\cos^2\Theta(t) - 1] \times \cos^2[2\Phi(t)] + 3\ \cos^2\Theta(t) - 1\Big]$$
$$+2\ \sin^2[2\Theta(t)] \times [\cos^2[2\Phi(t)] + 3] \tag{145}$$

and

$$\frac{d^2H}{d\Phi^2(t)} = -8\ \cos[4\Phi(t)] \times [\cos^2\Theta(t) - 1]^2 \tag{146}$$

and

$$\frac{d^2H}{d\Theta(t)\ d\Phi(t)} = 4\ \sin[4\Phi(t)] \times \sin[2\Theta(t)] \times [\cos^2\Theta(t) - 1]. \tag{147}$$

We write

$$D = \frac{d^2H}{d\Theta^2(t)} \times \frac{d^2H}{d\Phi^2(t)} - [\frac{d^2H}{d\Theta(t)\ d\Phi(t)}]^2. \tag{148}$$

When

$$\cos^2\Theta(t) = 1, \qquad \text{for any} \qquad \Phi(t) \tag{149}$$

we find

$$D = 0 \tag{150}$$

which implies that the second derivative test is inconclusive.

When

$$\Phi(t) = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi, \qquad \text{and} \qquad \cos^2\Theta(t) = \frac{1}{2} \tag{151}$$

we find

$$D = -16\,\sin^2[2\Theta(t)] < 0 \tag{152}$$

which implies that it is saddle point.

When

$$\Phi(t) = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}, \qquad \text{and} \qquad \Theta(t) = \frac{\pi}{2},\ \frac{3\pi}{2} \tag{153}$$

we find

$$D = -32 < 0 \tag{154}$$

which implies that it is saddle point.

When

$$\Phi(t) = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}, \qquad \text{and} \qquad \cos^2\Theta(t) = \frac{1}{3} \tag{155}$$

we find

$$D = \frac{64}{3}\,\sin^2[2\Theta(t)] > 0, \qquad \frac{d^2H}{d\Theta^2(t)} = 6\,\sin^2[2\Theta(t)] > 0 \tag{156}$$

which gives the minimum value

$$H_{\min} = 0. \tag{157}$$

When

$$\Phi(t) = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi, \qquad \text{and} \qquad \Theta(t) = \frac{\pi}{2},\ \frac{3\pi}{2} \tag{158}$$

we find

$$D = 64 > 0, \qquad \frac{d^2H}{d\Theta^2(t)} = -8 < 0 \tag{159}$$

which gives the maximum value

$$H_{\max} = \frac{4}{3}. \tag{160}$$

Hence we find from eqs. (138), (157) and (160) that

$$0 \le \left[\frac{1}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)]\right]^2 + \left[\frac{1}{2\sqrt{3}} \times [1 - 3\cos^2\Theta(t)]\right]^2 \le \frac{1}{3}. \tag{161}$$

Using eq. (161) in eq. (136) we find

$$\frac{2}{3} \le |h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 \le 1. \tag{162}$$

Since each individual square terms in eq. (136) are positive we find from eqs. (136), (161) and (162) that we can write

$$|h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 = \sin^2\theta(t) \tag{163}$$

and

$$\left[\frac{1}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)]\right]^2 + \left[\frac{1}{2\sqrt{3}} \times [1 - 3\cos^2\Theta(t)]\right]^2 = \cos^2\theta(t) \tag{164}$$

where

$$\frac{2}{3} \le \sin^2\theta(t) \le 1. \tag{165}$$

Since $|h_{12}(t)|$, $|h_{13}(t)|$ and $|h_{23}(t)|$ are positive we write eq. (163) as

$$\begin{aligned}
|h_{12}(t)| &= \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \\
|h_{13}(t)| &= \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \\
|h_{23}(t)| &= \sin\theta(t) \times \cos\sigma(t)
\end{aligned} \tag{166}$$

where

$$\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \quad \le \quad \theta(t) \quad \le \quad \pi - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right), \qquad\qquad 0 \le \sigma(t),\ \eta(t) \le \frac{\pi}{2}. \tag{167}$$

We write eqs. (164) as

$$\begin{aligned}
&\frac{1}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)] = \cos\theta(t) \times \sin\phi(t), \\
&\frac{1}{2\sqrt{3}} \times [1 - 3\cos^2\Theta(t)] = \cos\theta(t) \times \cos\phi(t)
\end{aligned} \tag{168}$$

where

$$0 \leq \phi(t) \leq 2\pi. \tag{169}$$

From eqs. (166), (168), (133), (135), (167) and (169) we find that general form of eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) are given by

$$\begin{aligned}
q_1(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \cos\phi_{12}(t), \\
q_2(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \sin\phi_{12}(t), \\
q_3(t) &= g \times \cos\theta(t) \times \sin\phi(t) \\
q_4(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \cos\phi_{13}(t), \\
q_5(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \sin\phi_{13}(t), \\
q_6(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \cos\phi_{23}(t), \\
q_7(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \sin\phi_{23}(t), \\
q_8(t) &= g \times \cos\theta(t) \times \cos\phi(t)
\end{aligned} \tag{170}$$

which reproduces eq. (7) where

$$\begin{aligned}
&\sin^{-1}(\sqrt{\frac{2}{3}}) \;\leq\; \theta(t) \;\leq\; \pi - \sin^{-1}(\sqrt{\frac{2}{3}}), \qquad 0 \leq \sigma(t),\; \eta(t) \leq \frac{\pi}{2}, \\
&0 \;\leq\; \phi(t) \leq 2\pi, \qquad -\pi \leq \phi_{12}(t),\; \phi_{13}(t),\; \phi_{23}(t) \leq \pi.
\end{aligned} \tag{171}$$

Hence we find that the general form of eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) is given by eq. (170) where the ranges of real time dependent phase factors $\theta(t),\ \sigma(t),\ \eta(t),\ \phi(t),\ \phi_{12}(t),\ \phi_{13}(t),\ \phi_{23}(t)$ are given by eq. (171). Note that the time dependent real phase factors $\theta(t),\ \sigma(t),\ \eta(t),\ \phi(t),\ \phi_{12}(t),\ \phi_{13}(t),\ \phi_{23}(t)$ in eq. (170) can not be independent of time t because if they are independent of t then the Yang-Mills potential $A^{\mu a}(x)$ in eq. (94) reduces to Maxwell-like potential $A^\mu(x)$.

XI. CONCLUSION

In Maxwell theory the the constant electric charge e of the electron is consistent with the continuity equation $\partial_\mu j^\mu(x) = 0$ where $j^\mu(x)$ is the current density of the electron. However,

in Yang-Mills theory the Yang-Mills color current density $j^{\mu a}(x)$ of the quark satisfies the equation $D_\mu[A]j^{\mu a}(x) = 0$ which is not a continuity equation ($\partial_\mu j^{\mu a}(x) \neq 0$) which implies that the color charge of the quark is not constant. Since the charge density of a point particle can be obtained from the quantum wave function of that particle, one finds that the charge density of a point particle may depend on space-coordinate $\vec{x}$. However, since the charge of a point particle is obtained from the corresponding charge density after integrating over the entire (physically) allowed volume $V = \int d^3\vec{x}$, one finds that the charge of the point particle can not depend on space-coordinate $\vec{x}$. Since the color charge of the quark is not constant and it can not depend on space coordinate $\vec{x}$, one finds that the color charge $q^a(t)$ of the quark in Yang-Mills theory has to be time dependent. In this paper we have derived the general form of eight time dependent fundamental color charges $q^a(t)$ of a quark in Yang-Mills theory in SU(3) where $a = 1, 2, ...8$

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